On Geometric Continuity of Isophotes

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Abstract. It is a well known fact that we can deduce G^n continuous isophotes from a G^{n+1} continuous surface. This paper gives answer to the reverse problem: we deduce a G^{n+1} continuous surface from G^n continuous isophotes on the surface. We show how many families of isophotes we have to consider and what constraints apply. Furthermore we apply the geodesic curvature and the "thickness" of isophotes as a surface interrogation tool.

§1. Introduction

Isophotes are a widely used interrogation tool in the design of various surfaces. First introduced in [4], they provide both an impression of global shape features and information about the continuity of the surface.

A family of isophotes on a surface $\mathbf{x}(u,v)$ is defined by a light direction vector $\mathbf{r}$ ($\|\mathbf{r}\| = 1$). Then the isophotes are the equipotential lines of the scalar field

$$s(u,v) = \mathbf{r} \cdot \mathbf{n}(u,v) \tag{1}$$

where $\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$ denotes the normalized normals of $\mathbf{x}$. This means, an isophote on the surface contains all surface points which have the same angle between the light direction and the surface normal. Silhouette lines are a special case of isophotes.

In this paper we use the following usual definition of geometric continuity: Two curves are G^n continuous at a common point $\mathbf{x}$ iff there exists a regular parametrization with respect to which they are C^n at $\mathbf{x}$. Two surfaces are G^n along a common line $\boldsymbol{l}$ iff there exists a regular parametrization with respect to which they are C^n along $\boldsymbol{l}$.

It is a well known fact that a G^{n+1} continuous surface implies G^n continuous isophotes (see [4] and [2]) . Section 3 of this paper gives answers to the reverse questions:

1) Is it possible to deduce G^{n+1} continuity of the surface from the G^n continuity of isophotes ?

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2) If so, how many families of isophotes do we have to consider, and what constraints apply ?

The answers to the questions 1) and 2) are quite important for using isophotes to analyze the continuity of surfaces. It shows how many families of isophotes have to be considered in order to get reliable statements about the continuity of the surface.

Isophotes can not generally be computed in a closed form but only as the numerical solution of partial differential equations. Nevertheless we want to compute local properties of isophotes, such as geodesic curvature and a new property called "thickness" in a closed form. In section 4, these properties are applied as surface interrogation methods.

Notation and abbreviations: $\mathbf{x}^{[i]}(t)$ denotes the i-th derivative vector of a parametrized curve $\mathbf{x}(t)$. $\mathbf{x}^{[i,j]}(u,v)$ denotes the partial derivative (i times in u-direction, j times in v-direction) of the parametrized surface $\mathbf{x}(u,v)$. For instance, $\mathbf{x}^{[2,1]}$ denotes $\mathbf{x}_{uuv}$. The partials $\mathbf{n}^{[i,j]}$ of the surface normals can be obtained by applying basic differentiation rules to $\mathbf{n}$. Furthermore, we use the classical abbreviations $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$, $G = \mathbf{x}_v \cdot \mathbf{x}_v$, $L = \mathbf{n} \cdot \mathbf{x}_{uu}$, $M = \mathbf{n} \cdot \mathbf{x}_{uv}$, $N = \mathbf{n} \cdot \mathbf{x}_{vv}$. From these scalar fields we can also compute the partial derivatives.

In this paper we only consider regularly parametrized curves and surfaces. This means for curves that $\|\dot{\mathbf{x}}(t)\| \neq 0$ for every t of the domain. For surfaces we assume that $\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{E \cdot G - F^2} \neq 0$.

§2. Theoretical Background

We will be analyzing isophotes on a parametric surface by interpreting them as tangent curves of vector fields. Before we discuss the surface case, we briefly describe the case of 2D vector fields.

Given is a 2D vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$. V assigns a vector $V(u,v) = (vx(u,v), vy(u,v))^T$ to any point (u,v) of the domain. A curve $\mathbf{t} \subseteq \mathbb{R}^2$ is called *tangent curve* (stream line, flow line, characteristic curve) of the vector field V if the following condition is satisfied: For all points $(u,v) \in \mathbf{t}$, the tangent vector of the curve in the point (u,v) has the same direction as the vector $V(u,v)$.

Tangent curves do dot depend on the magnitudes of the vectors in V but only on their directions. A point $(u,v) \in \mathbb{R}^2$ is called *critical point of* V if $V(u,v) = \mathbf{0}$ is the zero vector.

We consider a non-critical point (u_0, v_0) in the domain of V. Then we know that one and only one tangent curve $\mathbf{t}(t) = (u(t), v(t))$ passes through (u_0, v_0). We assume $\mathbf{t}(t_0) = (u_0, v_0)$. From the definition of tangent curves we know about the tangent vector of $\mathbf{t}$ in (u_0, v_0):

$$\dot{\mathbf{t}}(t_0) = \begin{pmatrix} \dot{u}(t_0) \\ \dot{v}(t_0) \end{pmatrix} = V(\mathbf{t}(t_0)) = \begin{pmatrix} vx(u_0, v_0) \\ vy(u_0, v_0) \end{pmatrix}. \tag{2}$$

Applying the chain rule to (2), we can compute the second derivative vector of $\mathbf{t}$ in (u_0, v_0):

$$\ddot{\mathbf{t}}(t_0) = (vx \cdot V_u + vy \cdot V_v)(u_0, v_0). \tag{3}$$

If we consider the domain of the vector field V as the domain of a surface $\mathbf{x}$ as well, the tangent curves of V are curves in the domain of $\mathbf{x}$ and therefore mapped onto surface curves on $\mathbf{x}$. Let $\mathbf{y}(t) = \mathbf{x}(\mathbf{t}(t))$ be the map of the tangent curve $\mathbf{t}(t)$ onto $\mathbf{x}$. Applying the chain rule to $\mathbf{x}(\mathbf{t}(t))$, we obtain for the tangent vectors of $\mathbf{y}$:

$$\begin{aligned}\dot{\mathbf{y}}(t_0) = \mathbf{y}^{[1]}(t_0) &= \mathbf{x}_u(\mathbf{t}(t_0)) \cdot \dot{u}(t_0) + \mathbf{x}_v(\mathbf{t}(t_0)) \cdot \dot{v}(t_0) \\ &= (vx \cdot \mathbf{x}_u + vy \cdot \mathbf{x}_v)(u_0, v_0).\end{aligned} \tag{4}$$

Defining

$$\begin{aligned}\mathbf{x}_0 &= \mathbf{x} \\ \mathbf{x}_{r+1} &= vx \cdot (\mathbf{x}_r)_u + vy \cdot (\mathbf{x}_r)_v \quad \text{for} \quad r = 0, 1, 2, ...\end{aligned} \tag{5}$$

we obtain for higher order derivatives of $\mathbf{y}$ in a similar way to (4):

$$\mathbf{y}^{[r]}(t_0) = \mathbf{x}_r(\mathbf{t}(t_0)) = \mathbf{x}_r(u_0, v_0) \quad \text{for} \quad r = 1, 2, 3, ... \quad . \tag{6}$$

A vector field defining the isophote directions in the domain of $\mathbf{x}$ is the perpendicular vector field to the gradient vector field of s defined in (1):

$$V(u,v) = \begin{pmatrix} vx(u,v) \\ vy(u,v) \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot \mathbf{n}_v(u,v) \\ \mathbf{r} \cdot \mathbf{n}_u(u,v) \end{pmatrix}. \tag{7}$$

The tangent curves of V are the isophotes in the domain, their maps onto $\mathbf{x}$ are the actual isophotes on the surface. Since

$$\begin{aligned}\mathbf{n}_u &= \frac{F \cdot M - G \cdot L}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2} \cdot \mathbf{x}_u + \frac{F \cdot L - E \cdot M}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2} \cdot \mathbf{x}_v \\ \mathbf{n}_v &= \frac{F \cdot N - G \cdot M}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2} \cdot \mathbf{x}_u + \frac{F \cdot M - E \cdot N}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2} \cdot \mathbf{x}_v\end{aligned}$$

we can write the isophotes vector field V in the domain as

$$V = \begin{pmatrix} vx \\ vy \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot (c \cdot \mathbf{x}_u + d \cdot \mathbf{x}_v) \\ \mathbf{r} \cdot (a \cdot \mathbf{x}_u + b \cdot \mathbf{x}_v) \end{pmatrix} \tag{8}$$

where

$$\begin{aligned} a = F \cdot M - G \cdot L \quad &, \quad b = F \cdot L - E \cdot M \\ c = F \cdot N - G \cdot M \quad &, \quad d = F \cdot M - E \cdot N. \end{aligned} \tag{9}$$

Critical points: occur where the isophotes vector field has a zero vector, i.e. $vx = 0$ and $vy = 0$. We obtain a critical point in $\mathbf{x}(u,v)$ iff at least one of the following conditions is satisfied:

- $\mathbf{r}$ is parallel to $\mathbf{n}(u,v)$,
- $\mathbf{x}(u,v)$ has a zero Gaussian curvature and $\mathbf{r}$ is in the plane defined by the normal and the principal direction with the zero normal curvature,
- $\mathbf{x}(u,v)$ is a flat point.

A proof of this can be found in [7].

§3. The Continuity of Isophotes

In this section we show how to infer a G^{n+1} surface from G^n isophotes. The result is formulated in theorem 2. To prove this we need the following

Lemma 1. *Given are two regularly parametrized curves* $\mathbf{x}(t)$ *and* $\tilde{\mathbf{x}}(t)$ *which join* C^n $(n > 0)$ *in the point* $\mathbf{x}_0 = \mathbf{x}(0) = \tilde{\mathbf{x}}(0)$*. Then the following statement is valid:* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *are* G^{n+1} *in* $\mathbf{x}_0$ *iff* $(\tilde{\mathbf{x}}^{[n+1]}(0) - \mathbf{x}^{[n+1]}(0))$ *is parallel to* $\mathbf{x}^{[1]}(0)$.

Proof: see [6]. ■

Now we can formulate the following

Theorem 2. *Given are two regularly parametrized surfaces* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *which join along a common line* $\boldsymbol{l}$*. Then* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *are* G^{n+1} *continuous* $(n \geq 1)$ *along* $\boldsymbol{l}$ *if there is one family of isophotes on* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *(defined by the direction vector* $\mathbf{r}$*) with the following properties:*

1) *In no point of* $\boldsymbol{l}$ *do the isophotes on* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *have critical points.*
2) *In no point of* $\boldsymbol{l}$ *are the isophotes on* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *tangent to* $\boldsymbol{l}$*.*
3) *In no point of* $\boldsymbol{l}$ *is the projection of* $\mathbf{r}$ *into the tangent plane of* $\mathbf{x}$ *and* $\tilde{\mathbf{x}}$ *tangent to* $\boldsymbol{l}$*.*
4) *All isophotes of the family are* G^n *continuous across* $\boldsymbol{l}$*.*

Proof: The direction vector $\mathbf{r}$ defines vx and vy with the values a, b, c, d on $\mathbf{x}$ (see (8) and (9)). In a similar way, $\mathbf{r}$ defines $\tilde{vx}$ and $\tilde{vy}$ with the values $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ on $\tilde{\mathbf{x}}$. We assume that the junction line $\boldsymbol{l}$ is $(0, v), 0 \leq v \leq 1$. This can be done by a linear reparametrization of $\mathbf{x}$ and $\tilde{\mathbf{x}}$ without loss of generality. Assumption 2) of the theorem can then be written in the form $vx(0, v) \neq 0$. We express $\mathbf{r}$ as $\mathbf{r} = q_1 \cdot \mathbf{x}_u + q_2 \cdot \mathbf{x}_v + q_3 \cdot \mathbf{n}$ where q_1, q_2 and q_3 are bivariate scalar functions over the domain of $\mathbf{x}$. Then assumption 3) of the theorem holds $q_1(0, v) \neq 0$. Since $q_1 \cdot (F^2 - E \cdot G) = (-G \cdot \mathbf{x}_u + F \cdot \mathbf{x}_v) \cdot (q_1 \cdot \mathbf{x}_u + q_2 \cdot \mathbf{x}_v + q_3 \cdot \mathbf{n})$ we obtain

$$(-G \cdot \mathbf{x}_u + F \cdot \mathbf{x}_v)(0, v) \cdot \mathbf{r} \neq 0. \tag{10}$$

The G^n continuity of the family of isophotes gives the G^n continuity of $\mathbf{x}$ and $\tilde{\mathbf{x}}$ along l. To show this, we can imagine a reparametrization of $\mathbf{x}$ and $\tilde{\mathbf{x}}$ in such a way that the isophotes defined by $\mathbf{r}$ are the isoparametric lines $v = \mathit{const}$ on $\mathbf{x}$ and $\tilde{\mathbf{x}}$. We thus can assume that $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are parametrized in such a way that they are C^n along $\boldsymbol{l}$. Since $\boldsymbol{l}$ is the isoparametric line $u = 0$, we can deduce $\mathbf{x}^{[i,j+1]}(0, v) = \tilde{\mathbf{x}}^{[i,j+1]}(0, v)$ from $\mathbf{x}^{[i,j]}(0, v) = \tilde{\mathbf{x}}^{[i,j]}(0, v)$. We obtain

$$\mathbf{x}^{[i,j]}(0, v) = \tilde{\mathbf{x}}^{[i,j]}(0, v) \quad \text{for} \quad i, j \in \mathbb{N}\ ,\ i + j \leq n + 1\ ,\ i \neq n + 1. \tag{11}$$

(9) and (11) yield along $\boldsymbol{l}$:

$$\begin{aligned}
c^{[i,j]} = \tilde{c}^{[i,j]}\ ,\ d^{[i,j]} = \tilde{d}^{[i,j]} \quad &\text{for} \quad i + j < n \\
a^{[i,j]} = \tilde{a}^{[i,j]}\ ,\ b^{[i,j]} = \tilde{b}^{[i,j]} \quad &\text{for} \quad i + j < n\ ,\ i \neq n - 1 \\
a^{[n-1,0]} - \tilde{a}^{[n-1,0]} &= (-G \cdot \mathbf{n} \cdot (\mathbf{x}^{[n+1,0]} - \tilde{\mathbf{x}}^{[n+1,0]})) \\
b^{[n-1,0]} - \tilde{b}^{[n-1,0]} &= (F \cdot \mathbf{n} \cdot (\mathbf{x}^{[n+1,0]} - \tilde{\mathbf{x}}^{[n+1,0]})).
\end{aligned} \tag{12}$$

From (8) and (12) we obtain along $\boldsymbol{l}$

$$\begin{aligned} vx^{[i,j]} &= \tilde{vx}^{[i,j]} \quad \text{for} \quad i+j<n \\ vy^{[i,j]} &= \tilde{vy}^{[i,j]} \quad \text{for} \quad i+j<n\ ,\ i\neq n-1 \\ vy^{[n-1,0]} - \tilde{vy}^{[n-1,0]} &= \mathbf{r}\cdot\left[(a^{[n-1,0]}-\tilde{a}^{[n-1,0]})\cdot\mathbf{x}_u + (b^{[n-1,0]}-\tilde{b}^{[n-1,0]})\cdot\mathbf{x}_v\right] \\ &= (\mathbf{n}\cdot(\mathbf{x}^{[n+1,0]}-\tilde{\mathbf{x}}^{[n+1,0]}))\cdot(\mathbf{r}\cdot(-G\cdot\mathbf{x}_u + F\cdot\mathbf{x}_v)). \end{aligned} \tag{13}$$

Let $\mathbf{y}^{[1]}(u,v)$ and $\tilde{\mathbf{y}}^{[1]}(u,v)$ be the tangent vectors of the isophotes on $\mathbf{x}$ and $\tilde{\mathbf{x}}$. From (5), (6) and (13) we obtain

$$\begin{aligned} \mathbf{y}^{[i]}(0,v) &= \tilde{\mathbf{y}}^{[i]}(0,v) \quad \text{for} \quad i\leq n-1 \\ (\mathbf{y}^{[n]}-\tilde{\mathbf{y}}^{[n]})(0,v) &= (vx^{n-1}\cdot(vy^{[n-1,0]}-\tilde{vy}^{[n-1,0]})\cdot\mathbf{x}_v)(0,v). \end{aligned} \tag{14}$$

(14) yields that the family of isophotes is C^{n-1} across $\boldsymbol{l}$. To achieve G^n of the isophotes we must have (see lemma 1):

$$(\mathbf{y}^{[n]}-\tilde{\mathbf{y}}^{[n]})(0,v) \quad \text{parallel to} \quad (vx\cdot\mathbf{x}_u + vy\cdot\mathbf{x}_v)(0,v). \tag{15}$$

(14), $vx(0,v)\neq 0$ and the assumption that $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are regularly parametrized yield the necessary condition for G^n of the isophotes across $\boldsymbol{l}$, i.e. for (15):

$$(vy^{[n-1,0]}-\tilde{vy}^{[n-1,0]})(0,v)=0. \tag{16}$$

Inserting (13) into (16) and keeping (10) in mind yields

$$\mathbf{n}(0,v)\cdot(\mathbf{x}^{[n+1,0]}-\tilde{\mathbf{x}}^{[n+1,0]})(0,v)=0. \tag{17}$$

Because of (17), there exist two scalar functions $p_1(v)$ and $p_2(v)$ so that

$$\tilde{\mathbf{x}}^{[n+1,0]}(0,v)=\mathbf{x}^{[n+1,0]}(0,v)+p_1(v)\cdot\mathbf{x}_u(0,v)+p_2(v)\cdot\mathbf{x}_v(0,v). \tag{18}$$

We consider the reparametrization $\hat{\mathbf{x}}$ of $\mathbf{x}$ which is defined as

$$\begin{aligned} \hat{\mathbf{x}}(u,v) &= \mathbf{x}(\hat{u}(u,v),\hat{v}(u,v)) \\ \hat{u}(u,v)=u+\frac{u^{n+1}}{(n+1)!}\cdot p_1(v) \quad &, \quad \hat{v}(u,v)=v+\frac{u^{n+1}}{(n+1)!}\cdot p_2(v). \end{aligned} \tag{19}$$

Computing the u-partials of $\hat{\mathbf{x}}$ by applying the chain rule to (19) yields for $u=0$:

$$\begin{aligned} \hat{\mathbf{x}}^{[i,0]}(0,v) &= \mathbf{x}^{[i,0]}(0,v)=\tilde{\mathbf{x}}^{[i,0]}(0,v) \quad \text{for} \quad 0\leq i\leq n \\ \hat{\mathbf{x}}^{[n+1,0]}(0,v) &= \mathbf{x}^{[n+1,0]}(0,v)+p_1(v)\cdot\mathbf{x}_u(0,v)+p_2(v)\cdot\mathbf{x}_v(0,v). \end{aligned} \tag{20}$$

From (18) and (20) we see that $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ are C^{n+1} along $\boldsymbol{l}$. Since $\hat{\mathbf{x}}$ is obtained from $\mathbf{x}$ by reparametrization, we have shown that $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are G^{n+1} along $\boldsymbol{l}$. ■

Remark: The special case $n=1$ of theorem 2 is already shown in [5]. The constraints there are formulated in a slightly different way but coincide with the constraints of theorem 2.

§4. Local Properties of Isophotes and Surface Interrogation

Since we were able to compute the first and second derivative vector of the isophote through a given surface point $\mathbf{x}(u_0, v_0)$ (see (6)), we can compute the geodesic curvature of the isophote in this point:

$$\begin{aligned}\ddot{\mathbf{y}}_p(t_0) &= \ddot{\mathbf{y}}(t_0) - (\mathbf{n}(u_0,v_0) \cdot \ddot{\mathbf{y}}(t_0)) \cdot \mathbf{n}(u_0,v_0) \\ \kappa(u_0,v_0) &= \text{sign}(\det[\dot{\mathbf{y}}(t_0), \ddot{\mathbf{y}}_p(t_0), \mathbf{n}(u_0,v_0)]) \cdot \frac{\|\dot{\mathbf{y}}(t_0) \times \ddot{\mathbf{y}}_p(t_0)\|}{\|\dot{\mathbf{y}}(t_0)\|^3}.\end{aligned} \tag{21}$$

$\ddot{\mathbf{y}}_p$ denotes the projection of $\ddot{\mathbf{y}}$ into the tangent plane. Since the geodesic curvature of a surface curve can be considered as the curvature of a 2D curve, it can be equipped with a sign.

The "thickness of isophotes" (or "distance between adjacent isophotes") is a measure of how strong the value of $s(u,v)$ changes locally. A strong change in s implies that "many isophotes are close together", one isophote is "thin". For the isophotes in the domain of $\mathbf{x}$ the measure of the "thickness" is $th = \frac{1}{\|\text{grad}(s)\|} = \frac{1}{\|V\|}$. Mapping this onto the surface, we obtain for the "thickness" of the isophotes through $\mathbf{x}(u_0, v_0)$:

$$th(u_0,v_0) = \frac{\|\mathbf{x}_u(u_0,v_0) \times \mathbf{x}_v(u_0,v_0)\|}{\|\dot{\mathbf{y}}(t_0)\|}. \tag{22}$$

Note that neither the geodesic curvature nor the "thickness" of the isophote through $\mathbf{x}(u_0, v_0)$ depends on the parametrization of $\mathbf{x}$. Also note that we were able to compute geodesic curvature and "thickness" of the isophote in $\mathbf{x}(u_0, v_0)$ in a closed form even if a closed form of the isophote itself does not exist.

Except for critical points of isophotes we can compute geodesic curvature and "thickness" of the isophotes for every surface point. Around critical points, geodesic curvature and "thickness" of isophotes diverge to infinity.

For using geodesic curvature and "thickness" as a surface interrogation method we compute and color code these measures for every surface point. For doing this we use a continuous color coding map with the following properties: a negative value gets a green color, a positive value gets a red color, the higher the magnitude of the value the lighter the color gets. In fact, a zero value gives black; if the value diverges to plus (minus) infinity the red (green) color tends to white.

The upper left picture of figure 1 shows the ray traced image of the shoe-shaped test surface. This surface consists of 29×10 piecewise bicubic patches and is G^2 continuous along the patch boundaries. The surface looks smooth, imperfections are hardly detectable.

The middle left picture shows the usual way of visualizing isophotes on the surface. The isophotes here are computed in the following way: choose a (small) interval and mark all points on the surface where the values of $s(u,v)$ are in the interval. The result are not the isophotes themselves but point sets

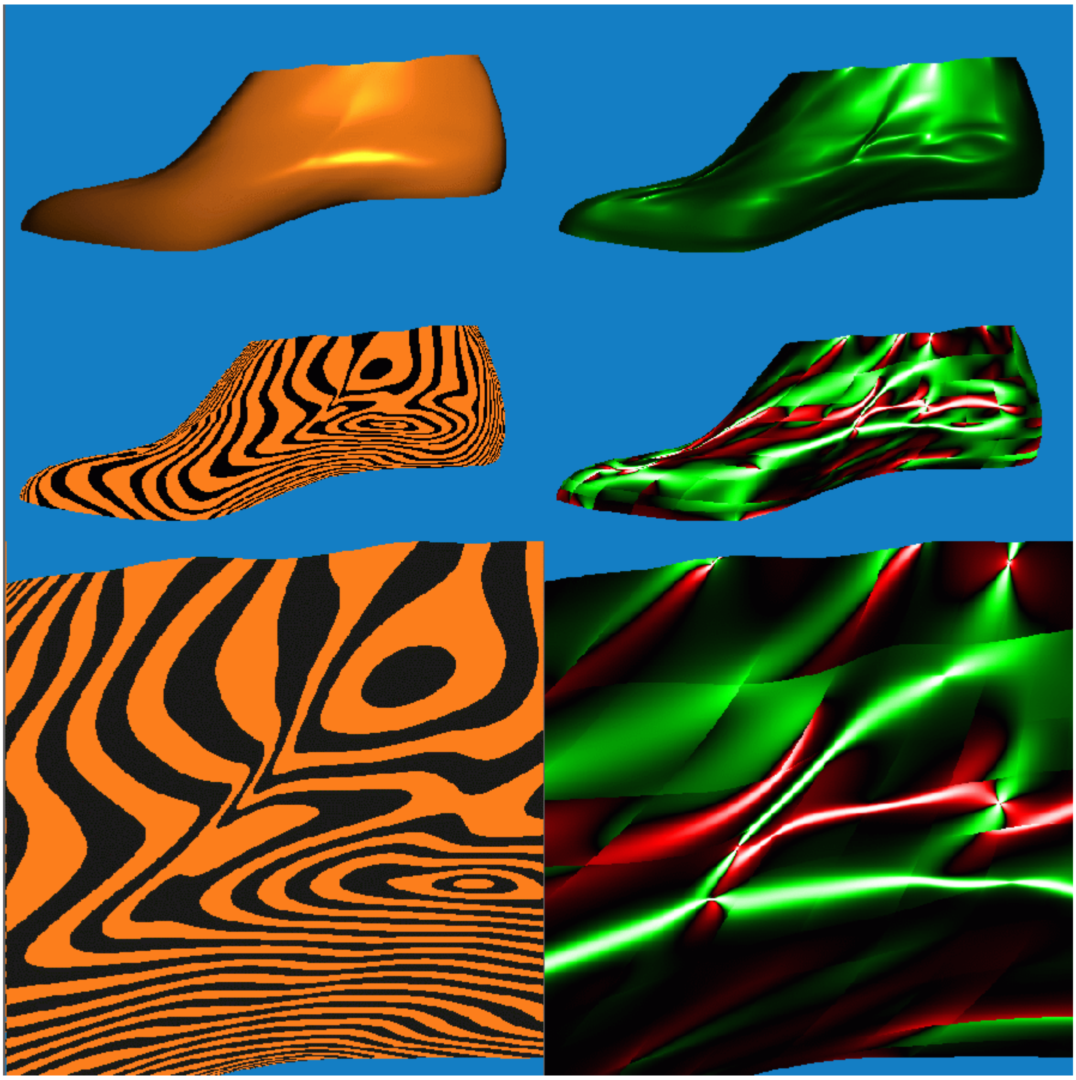

Fig. 1. Isophotes and their local properties on a test surface.

on the surface which give an impression of the behavior of the isophotes. In particular we can see that the point sets have a varying "thickness".

The upper right picture of figure 1 shows the visualization of the "thickness" of the isophotes. Here we clearly detect areas of the surface where a redesign is necessary. The critical points of isophotes appear as highlights in the visualization.

The middle right picture of figure 1 shows the visualization of the geodesic curvature of the isophotes. Again, the critical points of isophotes appear as highlights. We can clearly detect that the isophtes are not curvature (i.e. G^2) continuous at the patch boundaries. Therefore the surface is not G^3 continuous.

The lower left and the lower right pictures of figure 1 are magnifications of the middle left and the middle right picture.

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